

ON CORNER AVOIDANCE OF β -ADIC HALTON SEQUENCES

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ABSTRACT. We consider the corner avoiding property of s -dimensional β -adic Halton sequences. After extending this class of point sequences in an intuitive way, we show that the hyperbolic distance between each element of the sequence and the closest corner of $[0, 1]^s$ is $\mathcal{O}\left(\frac{1}{N^{s/2+\epsilon}}\right)$, where N denotes the index of the element. In our proof we use tools from Diophantine analysis, more precisely, we apply Schmidt's Subspace Theorem.

1. INTRODUCTION

In this article we consider special distributional properties of deterministic point sequences $(\mathbf{x}_n)_{n>0}$ in the s -dimensional unit cube. First, we define an s -dimensional interval $[\mathbf{a}, \mathbf{b}] \subseteq [0, 1]^s$ as $[\mathbf{a}, \mathbf{b}] = \{\mathbf{x} \in [0, 1]^s : a_i \leq x_i < b_i, i = 1, \dots, s\}$, where $\mathbf{x} = (x_1, \dots, x_s)$. Furthermore, we call a point sequence $(\mathbf{x}_n)_{n>0}$ uniformly distributed modulo 1 (u.d.), if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{[\mathbf{a}, \mathbf{b}]}(\mathbf{x}_n) = \lambda_s([\mathbf{a}, \mathbf{b}])$$

for all s -dimensional intervals $[\mathbf{a}, \mathbf{b}] \subseteq [0, 1]^s$, where λ_s denotes the s -dimensional Lebesgue measure. An equivalent characterization is given by a famous theorem of Weyl stating that a point sequence $(\mathbf{x}_n)_{n>0}$ is u.d. if and only if for every real-valued continuous function f the relation

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(\mathbf{x}_n) = \int_{[0, 1]^s} f(\mathbf{x}) d\mathbf{x}$$

holds. This result gives a hint how such sequences can be used for numerical integration, which is usually called Quasi-Monte Carlo (QMC) integration. However, it does not reveal anything about the size of the integration error. Fortunately, the Koksma-Hlawka inequality, see [10], states that

$$\left| \frac{1}{N} \sum_{n=1}^N f(\mathbf{x}_n) - \int_{[0, 1]^s} f(\mathbf{x}) d\mathbf{x} \right| \leq V(f) D_N^*(\mathbf{x}_n),$$

where the star-discrepancy D_N^* is defined as

$$D_N^*(\mathbf{x}_n) = D_N^*(\mathbf{x}_1, \dots, \mathbf{x}_N) = \sup_{\mathbf{a} \in [0, 1]^s} \left| \frac{1}{N} \sum_{n=1}^N \mathbf{1}_{[0, \mathbf{a}]}(\mathbf{x}_n) - \lambda_s([0, \mathbf{a}]) \right|$$

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and $V(f)$ denotes the variation of f in the sense of Hardy and Krause. By the fact that the best known QMC sequences, so-called low-discrepancy sequences, have an asymptotic star-discrepancy of order $\mathcal{O}\left(\frac{\log(N)^s}{N}\right)$, we get that the error of QMC integration converges for every s faster than the corresponding error of ordinary Monte Carlo integration, where random instead of deterministic point sequences are used. For a detailed discussion of uniformly distributed sequences and discrepancy, see e.g. [1].

Obviously, the Koksma-Hlawka inequality is useful only if $V(f) < \infty$, which further implies that f has no singularities in $[0, 1]^s$. Unfortunately, there are many potential applications of QMC integration, where the integrand function has a singularity, for instance the pricing of Asian options in the Black-Scholes model. For a detailed discussion we refer to [13]. However, in such cases an analogon of the Koksma-Hlawka inequality holds if the points of the sequence $(\mathbf{x}_n)_{n>0}$ stay sufficiently far away from the singularities and the integrand function satisfies mild regularity conditions. More precisely, we have the following theorem due to Owen [13]:

Theorem. *Let $f(\mathbf{x})$ be a real valued function on $[0, 1]^s$ which satisfies*

$$|\partial^u f(\mathbf{x})| \leq B \prod_{i=1}^s \left(x^{(i)}\right)^{-A_i - \mathbf{1}_{\{j \in u\}}} \quad \text{for all } u \subseteq \{1, \dots, s\},$$

where $\mathbf{x} = (x^{(1)}, \dots, x^{(s)})$, $A_i > 0$ and $B < \infty$. Suppose that $(\mathbf{x}_n)_{n>0}$ satisfies

$$\prod_{i=1}^s x_n^{(j)} \geq cN^{-r} \quad (1)$$

for all $1 \leq n \leq N$. Then for any $\eta > 0$

$$\left| \frac{1}{N} \sum_{n=1}^N f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x} \right| \leq C_1 D_N^*(\mathbf{x}_n) N^{\eta + r \max_i A_i} + C_2 N^{r(\max_i A_i - 1)}$$

holds for constants C_1 and C_2 which may depend on η .

For our purposes it is sufficient to consider only singularities in the corners of the s -dimensional unit cube, since if there is only a finite number of singularities, every integrand function can be transformed appropriately. In the sequel, we say $(\mathbf{x}_n)_{n>0}$ is (hyperbolically) avoiding the origin, if (1) holds. Owen [13] presents similar findings in case that singularities in the origin and in the corner $\mathbf{1} = (1, \dots, 1)$ have to be avoided simultaneously. These results have been extended by Hartinger et al. [8] who give bounds for the integration error, when there are singularities in all corners of the unit cube and the underlying point sequences avoid all corners hyperbolically.

The corner avoidance property has been investigated for many low-discrepancy sequences. First results are due to Sobol [17] and Owen [13], who investigate the origin avoiding property of Sobol and Halton sequences. In [8], Hartinger et al. investigate Halton, Faure and generalized Niederreiter sequences and their hyperbolic distance to all corners of the unit hypercube, where they compute the optimal values for r in (1). Hartinger and Ziegler [9] extend these results to random start Halton-sequences.

In the present article we consider the β -adic Halton sequence, introduced in [11], which is a natural extension of Halton's original construction. More precisely, we

consider numeration systems $G = (G_n)_{n \geq 0}$, with base sequence G_n constructed by a linear recurrence of length d , i.e.

$$G_{n+d} = a_0 G_{n+d-1} + \dots + a_{d-1} G_n,$$

where $d \geq 1$, $G_0 = 1$, $G_k = a_0 G_{k-1} + \dots + a_{k-1} G_0 + 1$ for $k < d$ and $a_i \in \mathbb{N}_0$ for $i = 0, \dots, d-1$. Then every positive integer n can be represented as

$$n = \sum_{k=0}^{\infty} \varepsilon_k(n) G_k,$$

where $\varepsilon_k(n) \in \{0, \dots, \lfloor G_{k+1}/G_k \rfloor - 1\}$ and $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$. This expansion (called G -expansion) is uniquely determined provided that

$$\sum_{k=0}^{K-1} \varepsilon_k(n) G_k < G_K, \quad (2)$$

where the digits $\varepsilon_k(n)$ are computed by the greedy algorithm (see for instance [4]). We call a sequence of digits $(\varepsilon_0, \varepsilon_1, \dots)$ regular if it satisfies (2) for every K . From now on we assume that all G -expansions are regular.

Let

$$X^d - a_0 X^{d-1} - \dots - a_{d-1} \quad (3)$$

be the characteristic polynomial of the numeration system $G = (G_n)_{n \geq 0}$. In this article we are interested only in numeration systems, where the characteristic polynomial is irreducible and the dominant root β of the characteristic polynomial (3) is a Pisot number. In this case we call β , which plays a crucial role, the characteristic root of the numeration system G . Note that, by [5, Theorem 2], the dominant root β of (3) is always a Pisot number if

$$a_0 \geq \dots \geq a_{d-1} \geq 1.$$

In the sequel we write $\mathbf{a} = (a_0, \dots, a_{d-1})$.

For an arbitrary integer n with (regular) G -expansion

$$n = \sum_{k=0}^{\infty} \varepsilon_k(n) G_k,$$

the so-called β -adic Monna map $\phi_\beta: \mathbb{N} \rightarrow \mathbb{R}$ is given as

$$\phi_\beta(n) = \phi_\beta \left(\sum_{j \geq 0} \varepsilon_j(n) G_j \right) = \sum_{j \geq 0} \varepsilon_j(n) \beta^{-j-1},$$

where β is the characteristic root of G . If we restrict ϕ_β to the set of representations which only have a finite number of non-zero entries, so-called admissible representations, we can define the so-called pseudo-inverse Monna map $\phi_\beta(n)^{-1}$, for more details see [11]. Note that if $d = 1$, then $(\phi_\beta(n))_{n > 0} = (\phi_{a_0}(n))_{n > 0}$ is the classical van der Corput sequence in base a_0 .

We call $Z(\varepsilon_0, \dots, \varepsilon_{k-1})$ a cylinder set of length k , where $Z(\varepsilon_0, \dots, \varepsilon_{k-1})$ is defined as the set of all admissible representations $(\varepsilon'_0, \varepsilon'_1, \dots)$ for which $\varepsilon_i = \varepsilon'_i$ for $0 \leq i \leq k-1$. We will sometimes write Z for short, if the context is clear.

In [11, Lemma 1], the authors prove that $\phi_\beta(\mathbb{N})$ is a dense subset of $[0, 1]$ if and only if the coefficients of G are of the form

$$\begin{aligned} \mathbf{a} &= (a_0, \dots, a_0) \quad \text{or} \\ \mathbf{a} &= (a_0, a_0 - 1, \dots, a_0 - 1, a_0), \end{aligned}$$

for $a_0 > 0$. Furthermore, they show that the β -adic Monna map transports the Haar measure on G to the Lebesgue measure on the unit interval. These results imply that there is an isometry between the odometer on the numeration system G and the dynamical system defined through the transformation $T_\beta: [0, 1] \rightarrow [0, 1]$, defined as $T_\beta \phi_\beta(n) = \phi_\beta(n + 1)$. Furthermore, assuming certain number theoretical conditions, the s -dimensional β -adic Halton sequence, with $\beta = (\beta_1, \dots, \beta_s)$, given as $(\phi_\beta(n))_{n>0} = (\phi_{\beta_1}(n), \dots, \phi_{\beta_s}(n))_{n>0}$ is uniformly distributed in $[0, 1]^s$, see [11]. The connection between ergodic theory and dynamical systems is drawn by Birkhoff's ergodic theorem, for a detailed discussion of this matter we refer to [6].

A further main ingredient for proving uniform distribution of the β -adic Halton sequence is [7, Theorem 5] which states that the odometer on a linear recurring numeration system G is uniquely ergodic and the corresponding invariant measure μ is given by

$$\mu(Z) = \frac{F_{K,0}\beta^{d-1} + (F_{K,1} - a_0 F_{K,0})\beta^{d-2} + \dots + (F_{K,d-1} - a_0 F_{K,d-2} - \dots - a_{d-2} F_{K,0})}{\beta^K(\beta^{d-1} + \beta^{d-2} + \dots + 1)}, \quad (4)$$

where $F_{K,r} := \#\{n < G_{K+r} : (\varepsilon_0(n), \varepsilon_1(n), \dots) \in Z\}$ and Z is a cylinder set of length K . We omit a detailed introduction of the ergodic properties of odometers on numeration systems and refer to [7].

The remainder of this article is structured as follows: In the next section we generalize some results presented in [11] to numeration systems with decreasing coefficients by extending the definition of the β -adic Monna map. In Section 3 we discuss the corner avoiding property for the extended β -adic Halton sequence. This property is proved by using the Subspace Theorem. Therefore we establish in Section 4 several auxiliary results that will enable us to prove this property for β -adic Halton sequence in the final section (Section 5).

2. EXTENDED β -ADIC HALTON SEQUENCES

In this section we slightly extend the definition of β -adic Halton sequences given in [11]. Using ergodic theory we prove that these sequences are uniformly distributed in $[0, 1]^s$. In the sequel, we denote by $\mathbf{a} = (a_0, \dots, a_{d-1})$ the coefficients of the numeration system G and we assume that $a_0 \geq a_1 \geq \dots \geq a_{d-1} \geq 1$.

Definition 1 (Extended β -adic Monna map). *Let G be a numeration system with characteristic root β and let the G -expansion of an integer n be given by*

$$n = \sum_{k=0}^{\infty} \varepsilon_k(n) G_k.$$

Then the extended β -adic Monna map $\psi_\beta: \mathbb{N} \rightarrow [0, 1]$ is defined as

$$\psi_\beta(n) = \sum_{k=0}^{\infty} f(n, k),$$

where

$$f(n, k) = \sum_{i=0}^{\varepsilon_k(n)-1} \mu(Z_n^k(i)), \quad (5)$$

μ is given in (4) and $Z_n^k(i)$ is the cylinder set with parameters $(\varepsilon_0(n), \dots, \varepsilon_{k-1}(n), i)$. As in [11], we can define a pseudo inverse of ψ_β , denoted by ψ_β^{-1} , by considering only digit representations with a finite number of non-zero entries.

Remark 1. Note that for a given integer n the cylinders involved in sums of type (5) are disjoint.

In order to illustrate the complexity of f , we give the following explicit example for $d = 3$ and $a_0 > a_1 > a_2 \geq 1$

$$f(n, k) = \begin{cases} \frac{\varepsilon_k(n)}{\beta^{k+1}}, & \text{if } \varepsilon_k(n) < a_2, \\ \frac{a_2}{\beta^{k+1}} + \frac{(\varepsilon_k(n) - a_2)(\beta^2 + \beta)}{\beta^{k+1}(\beta^2 + \beta + 1)}, & \text{if } \varepsilon_k(n) < a_1 \vee (\varepsilon_k(n) \leq a_1 \wedge k=1) \\ \frac{a_2}{\beta^{k+1}} + \frac{(a_1 + 1 - a_2)(\beta^2 + \beta)}{\beta^{k+1}(\beta^2 + \beta + 1)} + \frac{(\varepsilon_k(n) - a_1 - 1)\beta^2}{\beta^{k+1}(\beta^2 + \beta + 1)}, & \text{if } (\varepsilon_k(n) > a_1 \wedge k=1) \\ \frac{a_2}{\beta^{k+1}} + \frac{(a_1 - a_2)(\beta^2 + \beta)}{\beta^{k+1}(\beta^2 + \beta + 1)} + \frac{(\varepsilon_k(n) - a_1)\beta^2}{\beta^{k+1}(\beta^2 + \beta + 1)}, & \text{otherwise.} \end{cases}$$

Lemma 1. Let G be a numeration system with characteristic root β . Then, $\psi_\beta(\mathbb{N})$ is a dense subset of $[0, 1)$. Furthermore we have

$$\mu(Z) = \lambda(\psi_\beta(Z)),$$

for all cylinder sets Z , where μ is given in (4) and λ is the one-dimensional Lebesgue measure.

Proof. By definition it follows that the μ -measure of the union of all cylinder sets is 1 thus $0 \leq \psi_\beta(n) < 1$ for all $n \in \mathbb{N}$. We want to emphasize that the intersections of the cylinder sets appearing in (5) are of measure 0. Let $x \in [0, 1)$, then by the definition of ψ_β , there exists a sequence Z_n^x of cylinder sets with digits $(\varepsilon_0, \dots, \varepsilon_{n-1})$, such that $x \in Z_n^x$ for every n . Furthermore, we define an increasing sequence of integers k_n as

$$k_n = \sum_{i=0}^{n-1} \varepsilon_i G_i.$$

Since $\lim_{n \rightarrow \infty} \mu(Z_n^x) = 0$, we have that $\lim_{n \rightarrow \infty} \psi_\beta(k_n) = x$. Finally it follows by construction that ψ_β is measure preserving (see also [7, Theorem 5]). \square

The following result generalizes Theorems 2 and 3 in [11].

Theorem 1. Let $G^{(1)}, \dots, G^{(s)}$ be numeration systems defined by

$$G_{n+d_i}^{(i)} = a_0^{(i)} G_{n+d_i-1}^{(i)} + \dots + a_{d_i-1}^{(i)} G_n^{(i)}$$

where $a_0^{(i)} \geq a_1^{(i)} \geq \dots \geq a_{d_i-1}^{(i)} \geq 1$ holds for $i = 1, \dots, s$. Furthermore let $\gcd(a_0^{(i)}, \dots, a_{d_i-1}^{(i)}, a_0^{(j)}, \dots, a_{d_j-1}^{(j)}) = 1$ for all $i \neq j$ and let $\frac{(\beta^{(i)})^k}{(\beta^{(j)})^l} \notin \mathbb{Q}$, for all $l, k \in \mathbb{N}$, where $\beta = (\beta^{(1)}, \dots, \beta^{(s)})$ denotes the s -tuple of characteristic roots of the numeration systems $G^{(1)}, \dots, G^{(s)}$. Then, the extended s -dimensional β -adic Halton sequence

$$(\psi_\beta(n))_{n \geq 0} = (\psi_{\beta^{(1)}}(n), \dots, \psi_{\beta^{(s)}}(n))_{n \geq 0}$$

is uniformly distributed in $[0, 1]^s$.

Proof. Since $a_0^{(i)} \geq a_1^{(i)} \geq \dots \geq a_{d_i-1}^{(i)} \geq 1$ we know by [18, Theorem 4.1] that the odometer on the base system $G^{(i)}$ has a purely discrete spectrum, which is given by

$$\Gamma_j = \{z \in \mathbb{C} : \lim_{n \rightarrow \infty} z^{G_n^{(j)}} = 1\}.$$

By a standard result from ergodic theory (see e.g. [6]), the Cartesian product system constructed by the odometers on the numeration systems $G^{(1)}, \dots, G^{(s)}$ is ergodic if and only if the discrete parts of the spectra intersect only at 1.

As mentioned in [7],

$$\lim_{n \rightarrow \infty} \frac{G_n^{(j)}}{(\beta^{(j)})^n} = b_j,$$

where the constant b_j can be computed by residue calculus. Using \sim for asymptotic equality (if $n \rightarrow \infty$) we obtain for fixed $l \in \mathbb{N}$

$$\begin{aligned} \exp\left(2\pi i \frac{G_n^{(j)}}{(\beta^{(j)})^l}\right) &\sim \exp\left(2\pi i b_j (\beta^{(j)})^{n-l}\right) \\ &\sim \exp\left(2\pi i G_{n-l}^{(j)}\right), \end{aligned}$$

and thus

$$\lim_{n \rightarrow \infty} \exp\left(2\pi i \frac{G_n^{(j)}}{(\beta^{(j)})^l}\right) = \lim_{n \rightarrow \infty} \exp\left(2\pi i G_{n-l}^{(j)}\right) = 1.$$

Now assume that $C_j = \gcd(a_0^{(j)}, \dots, a_{d_j-1}^{(j)})$, then for every $k \in \mathbb{N}$ there exists an integer n_0 with $C_j^k \mid G_n^{(j)}$ for all $n \geq n_0$ and there exist no integers $C', n'_0 \in \mathbb{N}$ with $\gcd(C', C_j) = 1$ such that $C' \mid G_n^{(j)}$ for all $n \geq n'_0$. Thus Γ_j can be written as

$$\Gamma_j = \left\{ \exp\left(2\pi i \frac{c}{C_j^m (\beta^{(j)})^l}\right) : m, l, c \in \mathbb{N} \cup \{0\} \right\}.$$

By the assumptions of the theorem the spectra only intersect at 1. Hence the product dynamical system is ergodic. Moreover, the unique ergodicity of the product system follows since the product dynamical system is isomorphic to a group rotation and therefore purely discrete.

Now it remains to show that $(\psi_{\beta}(n))_{n \geq 0}$ is uniformly distributed in $[0, 1]^s$. We follow the ideas given in [11]. Lemma 1 implies that the extended β -adic Monna map ψ_{β} transports the measure μ on the system of G -expansions to the Lebesgue measure on the unit interval. Furthermore, by Lemma 1 the Monna map ψ_{β} , together with its pseudo inverse, defines an isomorphism, which implies that the dynamical system on $[0, 1]^s$ defined by the transformation $T_{\beta}: [0, 1]^s \rightarrow [0, 1]^s$, with $T_{\beta}(\psi_{\beta}(n)) = \psi_{\beta}(n+1)$ is uniquely ergodic. The proof is complete by applying Birkhoff's ergodic theorem. \square

3. CORNER AVOIDANCE

In this and the following sections we keep the assumptions and notations of the previous sections. In particular, we consider an s -dimensional β -adic Halton sequence $(\psi_{\beta}(n))_{n \geq 0}$ subject to s numeration systems $G^{(1)}, \dots, G^{(s)}$ with coefficients

$\mathbf{a}^{(i)} = (a_0^{(i)}, \dots, a_{d_i}^{(i)})$ and let us assume that $a_0^{(i)} = \dots = a_{k_i}^{(i)} > a_{k_i+1}^{(i)}$. As before we denote by $\beta^{(i)}$ the characteristic root of the numeration system $G^{(i)}$ for $1 \leq i \leq s$ and for $1 \leq j \leq d_i$ we denote by $\beta_j^{(i)}$ their conjugates. By convention we write $\beta_1^{(i)} = \beta^{(i)}$. Let $K^{(i)}$ be the Galois closure of $\mathbb{Q}(\beta^{(i)})$, i.e. $K^{(i)} = \mathbb{Q}(\beta_1^{(i)}, \dots, \beta_{d_i}^{(i)})$, and let $\Gamma^{(i)} = \text{Gal}(K^{(i)}/\mathbb{Q})$ be its Galois group.

Let $\mathbf{h} = (h^{(1)}, \dots, h^{(s)}) \in \{0, 1\}^s$ be a corner of the unit cube $[0, 1]^s$. We define the hyperbolic distance from $\mathbf{x} = (x^{(1)}, \dots, x^{(s)}) \in [0, 1]^s$ to the corner \mathbf{h} by

$$\|\mathbf{x}\|_{\mathbf{h}} = \prod_{i=0}^s |x^{(i)} - h^{(i)}|,$$

where $|\cdot|$ is the usual absolute value. For the rest of the paper we aim to prove the following theorem:

Theorem 2. *Assume that $K^{(i)} \cap K^{(j)} = \mathbb{Q}$ for all $1 \leq i < j \leq s$. Then the β -adic Halton sequence $(\psi_{\beta}(n))_{n \geq 0}$ avoids corners, i.e. for any $\varepsilon > 0$ there exists a constant $C_{\varepsilon, \beta}$ such that*

$$\|\psi_{\beta}(N)\|_{\mathbf{h}} > \frac{C_{\varepsilon, \beta}}{N^{H/2+\varepsilon}} \quad (6)$$

where

$$H = \begin{cases} 2 & \text{if } \mathbf{h} = (0, \dots, 0) \\ s & \text{if } \mathbf{h} = (1, \dots, 1) \\ 1 + \sum_{i=1}^s h^{(i)} & \text{in all other cases} \end{cases}.$$

In order to prove Theorem 2 we use Schmidt's famous Subspace Theorem [15]. More precisely we use a version of the Subspace Theorem that goes back to Schlickewei [14]. To formulate the Theorem properly we have to discuss the absolute values of a number field K/\mathbb{Q} with maximal order \mathfrak{o} . It is well known that for each prime ideal \mathfrak{p} in \mathfrak{o} there is a finite field $\mathbb{F} = \mathfrak{o}/\mathfrak{p}$ with $p^{n_{\mathfrak{p}}}$ elements, where $p > 0$ is the characteristic of \mathbb{F} . The number $n_{\mathfrak{p}}$ is called the local degree of \mathfrak{p} . We assign to each prime ideal \mathfrak{p} of \mathfrak{o} an absolute value $|\alpha|_{\mathfrak{p}} := p^{n_{\mathfrak{p}} v_{\mathfrak{p}}(\alpha)}$ on K , where $v_{\mathfrak{p}}(\alpha)$ is the unique integer such that $(\alpha) = \mathfrak{p}^{v_{\mathfrak{p}}(\alpha)} \mathfrak{q}$ and \mathfrak{p} and \mathfrak{q} are coprime fractional ideals. We call all these absolute values the finite or non-Archimedean canonical absolute values.

Similarly we define infinite absolute values. Therefore we note that a number field K/\mathbb{Q} of degree $d = [K : \mathbb{Q}]$ has d embeddings $\sigma : K \hookrightarrow \mathbb{C}$. If the embedding σ is real we put $n_{\sigma} = 1$ and put $n_{\sigma} = 2$ otherwise. For each embedding σ we define an absolute value on K by $|\alpha|_{\sigma} = |\sigma\alpha|^{n_{\sigma}}$ where $|\cdot|$ is the usual absolute value in \mathbb{C} . Note that with a complex embedding σ also its complex conjugate $\bar{\sigma}$ is an embedding inducing the same absolute value. We call all these absolute values $|\cdot|_{\sigma}$ the infinite or Archimedean canonical absolute values.

We denote by $M(K)$ the set of all canonical absolute values. The product formula (see e.g. [12, Chapter III, Theorem 1.3]) states that

$$\prod_{\nu \in M(K)} |\alpha|_{\nu} = 1. \quad (7)$$

for all algebraic numbers $\alpha \in K$. Note that the product is defined since $|\alpha|_{\nu} \neq 1$ for at most finitely many absolute values ν .

With these notations we are able to state the absolute Subspace Theorem in its simplest form suitable for our proof (cf. [16, Chapter V, Theorem 1D]).

Theorem 3 (Subspace Theorem). *Let K be an algebraic number field with maximal order \mathfrak{o} and let $S \subset M(K)$ be a finite set of absolute values which contains all of the Archimedean ones. For each $\nu \in S$ let $L_{\nu,1}, \dots, L_{\nu,n}$ be n linearly independent linear forms in n variables with coefficients in K . Then for given $\delta > 0$, the solutions of the inequality*

$$\prod_{\nu \in S} \prod_{i=1}^n |L_{\nu,i}(\mathbf{x})|_{\nu} < \overline{|\mathbf{x}|}^{-\delta}$$

with $\mathbf{x} = (x_1, \dots, x_n) \in \mathfrak{o}^n$ and $\mathbf{x} \neq \mathbf{0}$, where

$$\overline{|\mathbf{x}|} = \max \{|x_i|_{\nu} : 1 \leq i \leq n, \nu \in M(K), \nu \text{ Archimedean}\},$$

lie in finitely many proper subspaces of K^n .

Remark 2. We want to note that this version of the Subspace Theorem is not state of the art. More recent versions of the Subspace Theorem due to Evertse and Schlickewei [3] or Evertse and Ferretti [2] exist. These results have the extra feature that they yield a bound for the number of subspaces. However, it is not possible, even in principal, to determine these finitely many subspaces. Therefore an effective version of Theorem 2 cannot be achieved with the methods of this paper. Since an application of such a more recent version of Schmidt's Subspace Theorem is technical and yields no extra gain in view of Theorem 2 we use the cited, older and simpler version.

4. AUXILIARY RESULTS

This section has two aims. First, we want to relate the corner avoidance property to a Diophantine inequality to which we can apply the Subspace Theorem (Theorem 3) and second, we provide several estimates and relations which will be used in the next section.

A first step is to recognize for which n the extended Monna map ψ_{β} takes its extremal values. The minimal values are easy to describe but the maximal values depend on \mathbf{a} and in particular on the index k such that $a_0 = a_1 = \dots = a_k > a_{k+1}$.

Lemma 2. *If*

$$\psi_{\beta}(N) \leq \frac{\beta^{d-1}}{\beta^n(\beta^{d-1} + \beta^{d-2} + \dots + 1)},$$

then N has a regular G -expansion of the form

$$N = \varepsilon_n(N)G_n + \dots,$$

i.e. $\varepsilon_i(N) = 0$ for all $i = 0, 1, \dots, n-1$. If

$$1 - \psi_{\beta}(N) \leq \frac{\beta^{d-1}}{\beta^n(\beta^{d-1} + \beta^{d-2} + \dots + 1)},$$

then N has a regular G -expansion $N = \sum_i \varepsilon_i(N)G_i$, where the digits $\varepsilon_i(N)$ with $i < n$ satisfy $\varepsilon_i(N) = a_0$ if $k+1 \nmid i$ or $i = 0$ and $\varepsilon_i(N) = a_0 - 1$ otherwise.

Proof. As mentioned in [7, Proof of Lemma 3], for a cylinder set Z with K digits, the functions $F_{K,r}$ given in (4) follow the same recurrence relation as the base sequence G_n . Thus

$$(F_{K,j} - a_0 F_{K,j-1} - \dots - a_{j-1} F_{K,0}) \geq 0, \quad \text{for } 0 \leq j \leq d,$$

and hence

$$\mu(Z_j) \geq \frac{\beta^{d-1}}{\beta^j(\beta^{d-1} + \beta^{d-2} + \dots + 1)},$$

where Z_j is a cylinder set with at most j digits.

Let us assume that

$$\psi_\beta(N) \leq \frac{\beta^{d-1}}{\beta^n(\beta^{d-1} + \beta^{d-2} + \dots + 1)},$$

but N has regular G expansion

$$N = \varepsilon_k(N)G_k + \dots,$$

where $\varepsilon_k(N) \neq 0$ and $k < n$. Then we have

$$\psi_\beta(N) \geq \mu(Z_k) \geq \frac{\beta^{d-1}}{\beta^k(\beta^{d-1} + \beta^{d-2} + \dots + 1)} > \frac{\beta^{d-1}}{\beta^n(\beta^{d-1} + \beta^{d-2} + \dots + 1)},$$

where $Z_k = Z(\overbrace{0, \dots, 0}^{k \text{ times}})$. Hence the first statement of the lemma is proved.

In order to prove the second statement we claim the following

Claim 1. *Let $N < G_n$ be an integer with regular G -expansion*

$$N = \varepsilon_0(N)G_0 + \dots + \varepsilon_{n-1}(N)G_{n-1}$$

which yields the word $w_N = \varepsilon_0(N) \dots \varepsilon_{n-1}(N)$. The word w_N is maximal with respect to lexicographic order if and only if $\varepsilon_i(N) = a_0$ for all indices i such that $k+1 \nmid i$ or $i = 0$ and $\varepsilon_i(N) = a_0 - 1$ for all other indices.

Proof of the claim. First, let us note that the digit expansion given in the claim is a regular G -expansion. We prove this by induction on n . Indeed for $n \leq k+1$ we have

$$G_n = a_0 + a_0 G_1 + \dots + a_0 G_{n-1} + 1 > a_0 + a_0 G_1 + \dots + a_0 G_{n-1}$$

and inequality (2) is satisfied. Now assume that all those expansions are regular for all $m < n$. We have to prove that also the expansion for n is regular, i.e. we have to show that inequality (2) holds for n , too. Therefore assume that $n = n'(k+1) + r$ with $r < k+1$. We obtain

$$\begin{aligned} a_0 + \dots + a_0 G_{n'(k+1)-1} + (a_0 - 1)G_{n'(k+1)} + a_0 G_{n'(k+1)+1} + \dots + a_0 G_n \\ < a_0 G_{n'(k+1)} + a_0 G_{n'(k+1)+1} + \dots + a_0 G_n \leq G_{n+1}, \end{aligned}$$

where the first inequality holds by the induction hypothesis with $m = n'(k+1) - 1$ and the second inequality holds because of the recursion defining the sequence G_n .

Since we assume that the G -expansion is regular no digit can be larger than a_0 . Therefore our construction for $n \leq k$ is optimal. Since

$$a_0 + \dots + a_0 G_{k+1} > G_{k+2}$$

we deduce that $\varepsilon_{k+1}(N) < a_0$. In general we have

$$\begin{aligned} G_{n'(k+1)+1} &= a_0 G_{n'(k+1)} + \dots + a_0 G_{(n'-1)(k+1)+1} + a_{k+1} G_{(n'-1)(k+1)} + \dots \\ &< a_0 G_{n'(k+1)} + \dots + a_0 G_{(n'-1)(k+1)+1} + a_0 G_{(n'-1)(k+1)}, \end{aligned}$$

hence all digits $\varepsilon_{(k+1)n'}(N)$ are less than a_0 . Hence our construction is optimal. \square

We continue the proof of Lemma 2. Since $1 = \sum \mu(Z)$, where the sum runs over all cylinder sets of length n , we deduce that

$$1 - \psi_\beta(N) \leq \frac{\beta^{d-1}}{\beta^n(\beta^{d-1} + \beta^{d-2} + \dots + 1)} \leq \mu(Z) \quad (8)$$

only if Z is the cylinder set of length n such that $\psi_\beta(Z)$ lies closest to the right edge. But, by the definition of ψ_β this is exactly the cylinder set Z of the form $Z(\varepsilon_0(N), \dots, \varepsilon_{n-1}(N))$ such that the corresponding word $w_N = \varepsilon_0(N) \dots \varepsilon_{n-1}(N)$ is maximal with respect to the lexicographic order. Therefore N satisfies inequality (8) provided its regular G -expansion lies in Z and by Claim 1 the proof is complete. \square

Let G be a numeration system as above and β its characteristic root and $\beta = \beta_1, \beta_2, \dots, \beta_d$ the conjugates of β . Then we have

$$G_n = \sum_{j=1}^d b_j \beta_j^n, \quad (9)$$

where $b_j \in K = \mathbb{Q}(\beta_1, \dots, \beta_d)$. Note that by our assumptions the characteristic polynomial is irreducible and has therefore no double zero.

Lemma 3. *The b_j are conjugates. In particular, if $\sigma \in \Gamma = \text{Gal}(K/\mathbb{Q})$ satisfies $\sigma(\beta_j) = \beta_k$ for some $1 \leq j, k \leq d$, then $\sigma(b_j) = b_k$. Moreover, there exists an algebraic integer D_0 such that all $D_0 b_j$ are algebraic integers. In fact D_0 can be chosen to be the discriminant of the order $\mathbb{Z}[\beta_1, \dots, \beta_d]$.*

Proof. Since the first d elements G_0, \dots, G_{d-1} of the recurrence $(G_n)_{n \geq 0}$ are given we obtain the linear system

$$G_j = b_1 \beta_1^j + \dots + b_d \beta_d^j \quad \text{for } 0 \leq j \leq d-1$$

in the unknowns b_1, \dots, b_d . It is easy to solve this linear system using Cramer's rule and we obtain

$$b_j = \frac{\begin{vmatrix} \beta_1^0 & \dots & \beta_{j-1}^0 & G_0 & \beta_{j+1}^0 & \dots & \beta_n^0 \\ \beta_1^1 & \dots & \beta_{j-1}^1 & G_1 & \beta_{j+1}^1 & \dots & \beta_n^1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_1^d & \dots & \beta_{j-1}^d & G_d & \beta_{j+1}^d & \dots & \beta_n^d \end{vmatrix}}{\begin{vmatrix} \beta_1^0 & \dots & \beta_{j-1}^0 & \beta_j^0 & \beta_{j+1}^0 & \dots & \beta_n^0 \\ \beta_1^1 & \dots & \beta_{j-1}^1 & \beta_j^1 & \beta_{j+1}^1 & \dots & \beta_n^1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_1^d & \dots & \beta_{j-1}^d & \beta_j^d & \beta_{j+1}^d & \dots & \beta_n^d \end{vmatrix}}. \quad (10)$$

If we apply σ to b_j , then the columns in the determinants of the numerator and denominator permute in the same way respectively. In the numerator the column containing the G_i with $0 \leq i \leq d-1$ go to the k -th column and the column with

the entries β_k^i for $0 \leq i \leq d-1$ is omitted, since the entries $\sigma^{-1}(\beta_k^i) = \beta_j^i$ with $0 \leq i \leq d-1$ do not occur in the numerator on the right side of (10). Therefore we obtain

$$\sigma(b_j) = \frac{\begin{vmatrix} \beta_1^0 & \cdots & \beta_{k-1}^0 & G_0 & \beta_{k+1}^0 & \cdots & \beta_n^0 \\ \beta_1^1 & \cdots & \beta_{k-1}^1 & G_1 & \beta_{k+1}^1 & \cdots & \beta_n^1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_1^d & \cdots & \beta_{k-1}^d & G_d & \beta_{k+1}^d & \cdots & \beta_n^d \end{vmatrix}}{\begin{vmatrix} \beta_1^0 & \cdots & \beta_{j-1}^0 & \beta_j^0 & \beta_{j+1}^0 & \cdots & \beta_n^0 \\ \beta_1^1 & \cdots & \beta_{j-1}^1 & \beta_j^1 & \beta_{j+1}^1 & \cdots & \beta_n^1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_1^d & \cdots & \beta_{j-1}^d & \beta_j^d & \beta_{j+1}^d & \cdots & \beta_n^d \end{vmatrix}} = b_k.$$

For the second statement note that the discriminant D_0 of the order $\mathbb{Z}[\beta_1, \dots, \beta_d]$ is defined as

$$D_0 = \begin{vmatrix} \beta_1^0 & \cdots & \beta_{j-1}^0 & \beta_j^0 & \beta_{j+1}^0 & \cdots & \beta_n^0 \\ \beta_1^1 & \cdots & \beta_{j-1}^1 & \beta_j^1 & \beta_{j+1}^1 & \cdots & \beta_n^1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_1^d & \cdots & \beta_{j-1}^d & \beta_j^d & \beta_{j+1}^d & \cdots & \beta_n^d \end{vmatrix}^2,$$

hence $D_0 b_1, \dots, D_0 b_d$ are algebraic integers and $D_0 \in \mathbb{Z}$. \square

Let us fix the corner $\mathbf{h} \in \{0, 1\}^s$ and let $I = \{i : h^{(i)} = 0\}$ and $J = \{j : h^{(j)} = 1\}$. In view of the hyperbolic distance and Lemma 2 we may assume that

$$N = \varepsilon_{n_i}^{(i)}(N)G_{n_i}^{(i)} + \cdots + \varepsilon_{N_i}^{(i)}(N)G_{N_i}^{(i)} \quad \forall i \in I \quad (11)$$

and

$$N = \varepsilon_0^{(j)}(N)G_0^{(j)} + \cdots + \varepsilon_{n_j}^{(j)}(N)G_{n_j}^{(j)} + \cdots + \varepsilon_{N_j}^{(j)}(N)G_{N_j}^{(j)} \quad \forall j \in J, \quad (12)$$

where for all $i < n_j$ the digits satisfy $\varepsilon_i^{(j)}(N) = a_0^{(j)}$ if $i \nmid k_j + 1$ or $i = 0$ and $\varepsilon_i^{(j)}(N) = a_0^{(j)} - 1$ otherwise. Then Lemma 2 yields

$$\|\psi_{\beta}(N)\|_{\mathbf{h}} > c_1 \prod_{i=1}^s |\beta^{(i)}|^{-n_i} \quad (13)$$

for some effective computable constant c_1 . For instance one can choose

$$c_1 = \prod_{i=1}^s \left(\frac{(\beta^{(i)})^{d_i-1}}{(\beta^{(i)})^{d_i-1} + (\beta^{(i)})^{d_i-2} + \cdots + 1} \right).$$

Thus in order to prove Theorem 2 it is enough to prove

$$\prod_{i=1}^s |\beta^{(i)}|^{-n_i} > \frac{C_{\epsilon, \beta}}{N^{H/2+\epsilon}}$$

with N subject to (11) and (12). Let us substitute into the equations (11) and (12) the explicit formula (9) for $G_n^{(i)}$. We obtain for $i \in I$

$$N = \sum_{j=1}^{d_i} x_j^{(i)} \quad \text{with} \quad x_j^{(i)} = \left(\beta_j^{(i)} \right)^{n_i} \overbrace{\sum_{k=0}^{N_i - n_i} b_j^{(i)} \epsilon_{n_i+k}^{(i)}(N) \left(\beta_j^{(i)} \right)^k}^{:= c_j^{(i)}} \quad \forall i \in I. \quad (14)$$

In case of $i \in J$ we want to decompose N into a sum similar to (14). Therefore let us compute

$$\begin{aligned}
N &= \sum_{j=1}^{d_i} \left(\sum_{n=0}^{n_i-1} \varepsilon_n^{(i)}(N) b_j^{(i)} \left(\beta_j^{(i)} \right)^n + \sum_{n=n_i}^{N_i} \varepsilon_n^{(i)}(N) b_j^{(i)} \left(\beta_j^{(i)} \right)^n \right) \\
&= \sum_{j=1}^{d_i} \left(\sum_{n=0}^{n_i-1} \varepsilon_n^{(i)}(N) b_j^{(i)} \left(\beta_j^{(i)} \right)^n + \left(\beta_j^{(i)} \right)^{n_i} \overbrace{\sum_{n=0}^{N_i-n_i} b_j^{(i)} \varepsilon_{n_i+n}^{(i)}(N) \left(\beta_j^{(i)} \right)^n}^{:= \tilde{c}_j^{(i)}} \right) \\
&= 1 + \sum_{j=1}^{d_i} \left(a_0 \sum_{n=0}^{n_i-1} b_j^{(i)} \left(\beta_j^{(i)} \right)^n - \sum_{n=0}^{\lfloor \frac{n_i-1}{k+1} \rfloor} b_j^{(i)} \left(\beta_j^{(i)} \right)^{n(k+1)} + \tilde{c}_j^{(i)} \left(\beta_j^{(i)} \right)^{n_i} \right) \\
&= 1 + \sum_{j=1}^{d_i} \left(b_j^{(i)} \frac{a_0 \left(1 - \left(\beta_j^{(i)} \right)^{n_i} \right)}{1 - \beta_j^{(i)}} - b_j^{(i)} \frac{1 - \left(\beta_j^{(i)} \right)^{(\lfloor \frac{n_i-1}{k+1} \rfloor + 1)(k+1)}}{1 - \left(\beta_j^{(i)} \right)^{k+1}} + \tilde{c}_j^{(i)} \left(\beta_j^{(i)} \right)^{n_i} \right) \\
&= 1 + \overbrace{\sum_{j=1}^{d_i} \frac{a_0 b_j^{(i)}}{1 - \beta_j^{(i)}} - \sum_{j=1}^{d_i} \frac{b_j^{(i)}}{1 - \left(\beta_j^{(i)} \right)^{k+1}}}^{:= A_i} \\
&\quad + \sum_{j=1}^{d_i} \left(\beta_j^{(i)} \right)^{n_i} \overbrace{\left(\tilde{c}_j^{(i)} - \frac{a_0 b_j^{(i)}}{1 - \beta_j^{(i)}} + \frac{b_j^{(i)} \left(\beta_j^{(i)} \right)^{(\lfloor \frac{n_i-1}{k+1} \rfloor + 1)(k+1) - n_i}}{1 - \left(\beta_j^{(i)} \right)^{k+1}} \right)}^{:= c_j^{(i)}} \\
&= A_i + \sum_{j=1}^{d_i} \left(\beta_j^{(i)} \right)^{n_i} c_j^{(i)}
\end{aligned}$$

Note that

$$0 \leq \left(\left\lfloor \frac{n_i-1}{k+1} \right\rfloor + 1 \right) (k+1) - n_i < k+1.$$

Therefore we obtain for $i \in J$ the equation $N - A_i = \sum_{j=1}^{d_i} x_j^{(i)}$, where $x_j^{(i)} = \left(\beta_j^{(i)} \right)^{n_i} c_j^{(i)}$. In order to unify this equation with equation (14) we put $A_i = 0$ for $i \in I$. Moreover, we define $\mathcal{A} = \{A_i : 1 \leq i \leq s\}$ and obtain

$$N - A_i = \sum_{j=1}^{d_i} x_j^{(i)} \quad (15)$$

for all $1 \leq i \leq s$. Further we define

$$I_a := \{i : A_i = a\} \quad \forall a \in \mathcal{A},$$

which yields a partition of the set $\{1, 2, \dots, s\}$ and we put $i_a = \min I_a$ for all $a \in \mathcal{A}$. Finally let us note that without loss of generality we may assume that $0 \in \mathcal{A}$. Since

otherwise we replace N by $N - a$ for some $a \in \mathcal{A}$. By such a change of N we change only the quantities A_i and $c_j^{(i)}$ for $1 \leq i \leq s$, $1 \leq j \leq d_j$. However the proof of Theorem 2 only depends on the size of these quantities but not on their specific form.

Remark 3. Let us emphasize here that inequality (6) in Theorem 2 can be replaced by

$$\|\psi_\beta(N)\|_{\mathbf{h}} > \frac{C_{\varepsilon, \beta}}{N^{\max\{2, |\mathcal{A}|\}/2 + \varepsilon}}. \quad (16)$$

Indeed we will prove this inequality which implies inequality (6) since obviously we have $H \geq \max\{2, |\mathcal{A}|\}$.

In our considerations below it will be important to estimate the quantities A_i for $1 \leq i \leq s$ and $c_j^{(i)}$ for $1 \leq i \leq s$, $2 \leq j \leq d_i$. Moreover let D be the common denominator of all $a \in \mathcal{A}$ and all $c_j^{(i)}$. Thus D is the smallest positive, rational integer D such that $Dc_j^{(i)}$ and Da are algebraic integers for all $1 \leq i \leq s$, $1 \leq j \leq d_i$ and $a \in \mathcal{A}$. Also an estimate of the quantity D will be needed in the course of the proof of Theorem 2. Therefore we prove the following lemma:

Lemma 4. *With the notations above we have $|c_j^{(i)}|, |A_i|, |D| \leq C$, provided $j \neq 1$, where C depends only on the s -tuple of numeration systems $(G^{(1)}, \dots, G^{(s)})$. Moreover, we have $A_i \in \mathbb{Q}$.*

Proof. First, note that $c_j^{(i)}$ in case of $i \in I$ and $\tilde{c}_j^{(i)}$ in case of $i \in J$ can be estimated for all $2 \leq j \leq d_i$ by a geometric series, hence both are bounded by a constant neither depending on N nor n_i . Moreover, for fixed i the common denominator is the integer D_0 from Lemma 3. But also the other components of $c_j^{(i)}$ in case of $i \in J$ only depend on $\beta_j^{(i)}$ and the constants $a_j^{(i)}$. Hence $|c_j^{(i)}| \leq C$ for all $1 \leq i \leq s$, $2 \leq j \leq d_i$ and a sufficiently large constant C . Also the common denominator of all $c_j^{(i)}$ is bounded by a constant neither depending on N nor on n_i . Similarly we obtain $|A_i| \leq C$ and the denominators of all A_i are again bounded by a constant. Hence, also $|D| \leq C$ for a sufficiently large constant C .

Therefore we are left to show that A_i is rational for all $1 \leq i \leq s$. But A_i can be written as the trace of some algebraic number $\alpha \in K^{(i)}$ and is therefore rational. Indeed we have $A_i = 0$ for all $i \in I_0$ and for all other indices i we have

$$\begin{aligned} A_i &= 1 + \sum_{j=1}^{d_i} \frac{a_0 b_j^{(i)}}{1 - \beta_j^{(i)}} - \sum_{j=1}^{d_i} \frac{b_j^{(i)}}{1 - (\beta_j^{(i)})^{k+1}} \\ &= \text{Tr}_{\mathbb{Q}(\beta^{(i)})/\mathbb{Q}} \left(\frac{1}{d_i} + \frac{a_0 b_1^{(i)}}{1 - \beta^{(i)}} - \frac{b_1^{(i)}}{1 - (\beta^{(i)})^{k+1}} \right) \in \mathbb{Q}. \end{aligned}$$

□

5. APPLICATION OF THE SUBSPACE THEOREM

A key step proving Theorem 2 is the following Proposition:

Proposition 1. *Let $a, b \in \mathcal{A}$ not necessarily distinct and $\epsilon > 0$. Then there exists a constant C_ϵ such that*

$$C_\epsilon N^{1+\epsilon} > \prod_{i \in I_a \cup I_b} \left| \beta_1^{(i)} \right|^{n_i}. \quad (17)$$

Once we have established (17) it is easy to deduce inequality (16) and hence Theorem 2. The case $|\mathcal{A}| \leq 2$ is a direct consequence of Proposition 1 and inequality (13). Therefore let us assume that $|\mathcal{A}| > 2$ and furthermore let us assume that $a_1, \dots, a_{|\mathcal{A}|}$ are the elements of \mathcal{A} in some order. Let us write $a_{|\mathcal{A}|+1} = a_1$ and $I_k = I_{a_k}$. Then according to Proposition 1 there exists a constant $C = C_{2\epsilon/|\mathcal{A}|}$ such that

$$\left(C N^{1+2\epsilon/|\mathcal{A}|} \right)^{|\mathcal{A}|} > \prod_{k=1}^{|\mathcal{A}|} \prod_{i \in I_k \cup I_{k+1}} \left| \beta_1^{(i)} \right|^{n_i} = \prod_{i=1}^s \left| \beta_1^{(i)} \right|^{2n_i} \geq \frac{c_1^2}{\|\psi_{\mathcal{B}}(N)\|_{\mathbf{h}}^2}$$

which immediately yields inequality (16).

Therefore our task is to prove Proposition 1, which takes the rest of the paper. Without loss of generality we may assume that $b = 0$ in Proposition 1 and we fix $a \in \mathcal{A}$. Moreover, we fix the number field K which is the compositum of the number fields $K^{(i)}$ with $i \in I_0 \cup I_a = I_{0,a}$. Further we denote by Γ the Galois group of K . Since we assume that all the fields $K^{(i)}$ are linearly disjoint we have $\Gamma = \prod_{i \in I_0 \cup I_a} \Gamma^{(i)}$.

In order to apply the Subspace Theorem, we have to determine which valuations are of particular interest in view of Theorem 2. Thus let

$$S_j^{(i)} = \left\{ \nu \in M(K) : \nu = |\cdot|_{\mathfrak{p}}, \mathfrak{p} \mid \left(\beta_j^{(i)} \right) \right\}$$

and

$$S_{\text{fin}} = \bigcup_{i \in I_{0,a}} \bigcup_{j=1}^{d_i} S_j^{(i)}$$

In particular, S_{fin} is the set of finite places of $M(K)$ that lie above all the primes that divide some principal ideal $\left(\beta_j^{(i)} \right)$ with $i \in I_{0,a}$ and $1 \leq j \leq d_i$. Moreover let

$$S_\infty = \{ \nu \in M(K) : \nu = |\cdot|_\sigma, \sigma \in \Gamma \}$$

be the set of all canonical Archimedean absolute values in K . In view of Theorem 3 we write $S = S_{\text{fin}} \cup S_\infty$.

Our next step is to construct linear forms $L_{\nu,j}^{(i)}$ for all $\nu \in S$. The aim is to find linear forms that match the conditions of the Subspace Theorem on the one hand and yield small values when they are evaluated at the point $\mathbf{x} = \left(x_j^{(i)} \right)$ on the other hand. Before we construct these linear forms we want to note that for $i \in I_a$ we have

$$x_1^{(i_a)} + \dots + x_{d_{i_a}}^{(i_a)} = N - a = x_1^{(i)} + \dots + x_{d_i}^{(i)}$$

provided $i \neq i_a$. Thus we can express $x_1^{(i)}$ as a linear combination of $x_j^{(i_a)}$ for $1 \leq j \leq d_{i_a}$ and $x_j^{(i)}$ for $2 \leq j \leq d_i$. Loosely speaking we can eliminate the unknown $x_1^{(i)}$ provided $i \neq i_a$ and $i \in I_a$. Therefore

$$X = \left\{ X_j^{(i)} : 1 \leq i \leq s, 2 \leq j \leq d_i \right\} \cup \left\{ X_1^{(i_0)}, X_1^{(i_a)} \right\}$$

is the set of indeterminates over which we construct our linear forms $L_{\nu,j}^{(i)} \in K[X]$.

5.1. Linear forms for non-Archimedean places. Let us assume that $\nu \in S_m^{(\ell)}$. Then we define:

- If $(i, j) = (i_a, 1)$ and $\ell \in I_a \setminus \{i_a\}$, then

$$L_{\nu,1}^{(i_a)}(X) = \left(X_1^{(i_a)} + \cdots + X_{d_{i_a}}^{(i_a)} \right) - \left(X_2^{(\ell)} + \cdots + X_{d_\ell}^{(\ell)} \right).$$

- and in all other cases

$$L_{\nu,j}^{(i)}(X) = X_j^{(i)}.$$

Note that for a fixed valuation $\nu \in S_m^{(\ell)}$ the linear forms $L_{\nu,j}^{(i)}(X)$ are linearly independent over K . Our next step is to compute

$$\prod_{\nu \in S_{\text{fin}}} \left| L_{\nu,j}^{(i)}(\mathbf{x}) \right|_\nu$$

for fixed i and j . Let us remind that by the product formula (7) we have

$$\prod_{\substack{\nu \in M(K) \\ \nu \text{ non-Archimedean}}} |\alpha|_\nu = \frac{1}{|N_{K/\mathbb{Q}}(\alpha)|}.$$

First, consider the case that (i, j) is not of the form $(i_a, 1)$. Since for all non-Archimedean absolute values

$$\left| L_{\nu,j}^{(i)}(\mathbf{x}) \right|_\nu = \left| x_j^{(i)} \right|_\nu = \left| c_j^{(i)} \left(\beta_j^{(i)} \right)^{n_i} \right|_\nu \leq \frac{\left| \left(\beta_j^{(i)} \right)^{n_i} \right|_\nu}{|D|_\nu}$$

we obtain

$$\prod_{\nu \in S_{\text{fin}}} \left| L_{\nu,j}^{(i)}(\mathbf{x}) \right|_\nu = \prod_{\nu \in S_{\text{fin}}} \left| x_j^{(i)} \right|_\nu \leq D^{|\Gamma|} \prod_{\nu \in S_{\text{fin}}} \left| \left(\beta_j^{(i)} \right)^{n_i} \right|_\nu \leq \frac{D^{|\Gamma|}}{\left| N_{K/\mathbb{Q}} \left(\left(\beta_j^{(i)} \right)^{n_i} \right) \right|}. \quad (18)$$

Now let us assume that $i = i_a$ and $j = 1$. Then we have

$$\left| L_{\nu,1}^{(i_a)}(\mathbf{x}) \right|_\nu = \left| \overbrace{\left(x_1^{(i_a)} + \cdots + x_{d_{i_a}}^{(i_a)} \right)}^{=N-a} - \overbrace{\left(x_2^{(\ell)} + \cdots + x_{d_\ell}^{(\ell)} \right)}^{=N-a-x_1^{(\ell)}} \right|_\nu = \left| x_1^{(\ell)} \right|_\nu$$

for $\nu \in S_m^{(\ell)}$ and $\ell \in I_a \setminus \{i_a\}$. Thus we obtain

$$\prod_{\nu \in S_{\text{fin}}} \left| L_{\nu,1}^{(i_a)}(\mathbf{x}) \right|_\nu \leq D^{|\Gamma|} \prod_{\ell \in I_a} \prod_{\nu \in S_1^{(\ell)}} \left| \beta_1^{(\ell)} \right|_\nu^{n_\ell} \leq \frac{|D|^{|\Gamma|}}{\prod_{\ell \in I_a} \left| N_{K/\mathbb{Q}} \left(\left(\beta_1^{(\ell)} \right)^{n_\ell} \right) \right|}. \quad (19)$$

5.2. Linear forms for Archimedean places. Now, let us turn to the Archimedean absolute values. As explained above every $\nu \in S_\infty$ corresponds to some $\sigma \in \Gamma$. Let us emphasize again that σ is unique up to complex conjugation, and every $\sigma \in \Gamma$ determines a $\nu \in S_\infty$. Let us assume for the moment that σ is a complex embedding. Then we also write $L_{\nu,j}^{(i)} = L_{\sigma,j}^{(i)}$ for some $\sigma \in \Gamma$. Note that $L_{\sigma,j}^{(i)} = L_{\bar{\sigma},j}^{(i)}$

for all $\sigma \in \Gamma$. Moreover, let us write $\|\cdot\|_\sigma = |\cdot|_\nu^{1/2}$ thus we have $|\cdot|_\nu = \|\cdot\|_\sigma \|\cdot\|_{\bar{\sigma}}$. Altogether we have with these notations

$$\prod_{\nu \in S_\infty} |L_{\nu,j}^{(i)}(\mathbf{x})|_\nu = \prod_{\sigma \in \Gamma} \|L_{\sigma,j}^{(i)}(\mathbf{x})\|_\sigma.$$

Now we have enough notations to define our linear forms. In case that $\sigma(\beta_j^{(i)}) \neq \beta_1^{(i)}$ we define

$$L_{\sigma,j}^{(i)}(X) = X_j^{(i)}.$$

In case that $\sigma(\beta_j^{(i)}) = \beta_1^{(i)}$ and

- if $i \in I_a$ and $i \neq i_a$, then we put

$$L_{\sigma,j}^{(i)}(X) = (X_1^{(i_a)} + \cdots + X_{d_{i_a}}^{(i_a)}) - (X_2^{(i)} + \cdots + X_{d_i}^{(i)}),$$

- if $i = i_a$ and $a \neq 0$, then we put

$$L_{\sigma,j}^{(i_a)}(X) = (X_1^{(i_0)} + \cdots + X_{d_{i_0}}^{(i_0)}) - (X_1^{(i_a)} + \cdots + X_{d_{i_a}}^{(i_a)}),$$

- and finally if $i = i_0$, then we put $L_{\sigma,j}^{(i_0)}(X) = X_j^{(i_0)}$.

Since by assumption $\beta_1^{(i)}$ is Pisot and therefore real, $\sigma(\beta_j^{(i)}) = \beta_1^{(i)}$ if and only if $\bar{\sigma}(\beta_j^{(i)}) = \beta_1^{(i)}$. Therefore the linear forms $L_{\nu,j}^{(i)}$ are well defined.

Our next task is to compute

$$\prod_j \prod_{\nu \in S} |L_{\nu,j}^{(i)}(\mathbf{x})|_\nu = \prod_j \prod_{\nu \in S_{\text{fin}}} |L_{\nu,j}^{(i)}(\mathbf{x})|_\nu \times \prod_j \prod_{\sigma \in \Gamma} \|L_{\sigma,j}^{(i)}(\mathbf{x})\|_\sigma$$

for fixed $i = 1, \dots, s$. Let us consider the case $i \neq i_0, i_a$, first. In this case we have $j > 1$. Let us assume that $\sigma(\beta_j^{(i)}) = \beta_m^{(i)}$ with $m > 1$, then

$$\|L_{\sigma,j}^{(i)}(\mathbf{x})\|_\sigma = \|x_j^{(i)}\|_\sigma = |c_m^{(i)} (\beta_m^{(i)})^{n_i}| = \mathcal{O}(|\beta_m^{(i)}|^{n_i})$$

due to Lemma 4. In case that $\sigma(\beta_j^{(i)}) = \beta_1^{(i)}$ and $i \in I_a \setminus \{i_a\}$ we have

$$\begin{aligned} \|L_{\sigma,j}^{(i)}(\mathbf{x})\|_\sigma &= \left\| \overbrace{(x_1^{(i_a)} + \cdots + x_{d_{i_a}}^{(i_a)})}^{=N-a} - \overbrace{(x_2^{(i)} + \cdots + x_{d_i}^{(i)})}^{=N-a-x_1^{(i)}} \right\|_\sigma \\ &= \|c_1^{(i)} (\beta_1^{(i)})^{n_i}\|_\sigma = \mathcal{O}(|\sigma(\beta_1^{(i)})|^{n_i}). \end{aligned}$$

Note that $\sigma(c_1^{(i)}) = c_{m'}^{(i)} = \mathcal{O}(1)$ since $m' \neq 1$ and in view of Lemma 4. With these estimates and inequality (18) we obtain

$$\begin{aligned}
& \prod_{j=2}^{d_i} \prod_{\nu \in S} \left| L_{\nu,j}^{(i)}(\mathbf{x}) \right|_{\nu} = \prod_{j=2}^{d_i} \prod_{\nu \in S_{\text{fin}}} \left| L_{\nu,j}^{(i)}(\mathbf{x}) \right|_{\nu} \\
& \quad \times \prod_{j=2}^{d_i} \prod_{k=2}^{d_i} \prod_{\substack{\sigma \in \Gamma \\ \sigma(\beta_j^{(i)}) = \beta_k^{(i)}}} \left\| L_{\sigma,j}^{(i)}(\mathbf{x}) \right\|_{\sigma} \times \prod_{j=2}^{d_i} \prod_{\substack{\sigma \in \Gamma \\ \sigma(\beta_j^{(i)}) = \beta_1^{(i)}}} \left\| L_{\sigma,j}^{(i)}(\mathbf{x}) \right\|_{\sigma} \\
& = \frac{D^{|\Gamma|}}{\left| N_{K/\mathbb{Q}} \left((\beta_1^{(i)})^{n_i} \right) \right|^{d_i-1}} \\
& \quad = \mathcal{O} \left(\frac{\left| N_{K/\mathbb{Q}} \left((\beta_1^{(i)})^{n_i} \right) \right|}{\left(|\beta_1^{(i)}|^{n_i} \right)^{\frac{|\Gamma|}{d_i}}} \right) \quad = \mathcal{O} \left(\prod_{k=2}^{d_i} \left(|\beta_k^{(i)}|^{n_i} \right)^{\frac{|\Gamma|}{d_i}} \right) \\
& \quad \times \prod_{j=2}^{d_i} \left(\prod_{m=2}^{d_i} \left(|c_m^{(i)}| \cdot |\beta_m^{(i)}|^{n_i} \right)^{\frac{|\Gamma|}{d_i}} \right) \times \prod_{j=2}^{d_i} \prod_{\substack{\sigma \in \Gamma \\ \sigma(\beta_j^{(i)}) = \beta_1^{(i)}}} \left| c_1^{(i)} \left(\beta_1^{(i)} \right)_i \right|_{\sigma} \\
& = \mathcal{O} \left(\frac{\left| N_{K/\mathbb{Q}} \left((\beta_1^{(i)})^{n_i} \right) \right|}{\left(|\beta_1^{(i)}|^{n_i} \right)^{|\Gamma|}} \right)
\end{aligned} \tag{20}$$

Next, let us consider the case $i = i_a$ but $a \neq 0$. Let $\sigma \in \Gamma$ be such that $\sigma(\beta_j^{(i_a)}) = \beta_1^{(i_a)}$. Then we use the following estimate

$$\left\| L_{\sigma,j}^{(i_a)}(\mathbf{x}) \right\|_{\sigma} = \left\| \overbrace{\left(x_1^{(i_a)} + \cdots + x_{d_{i_a}}^{(i_a)} \right)}^{=N-a} - \overbrace{\left(x_1^{(i_0)} + \cdots + x_{d_{i_0}}^{(i_0)} \right)}^{=N} \right\|_{\sigma} = \|a\|_{\sigma} = \mathcal{O}(1).$$

Thus applying (18) and (19) we get

$$\begin{aligned}
\prod_{j=1}^{d_{i_a}} \prod_{\nu \in S} \left| L_{\nu,j}^{(i_a)}(\mathbf{x}) \right|_{\nu} &= \prod_{j=1}^{d_{i_a}} \prod_{\nu \in S_{\text{fin}}} \left| L_{\nu,j}^{(i_a)}(\mathbf{x}) \right|_{\nu} \\
&\times \prod_{j=1}^{d_{i_a}} \prod_{m=2}^{d_{i_a}} \prod_{\substack{\sigma \in \Gamma \\ \sigma(\beta_j^{(i_a)}) = \beta_m^{(i_a)}}} \left\| L_{\sigma,j}^{(i_a)}(\mathbf{x}) \right\|_{\sigma} \times \overbrace{\prod_{j=1}^{d_{i_a}} \prod_{\substack{\sigma \in \Gamma \\ \sigma(\beta_j^{(i_a)}) = \beta_1^{(i_a)}}} \left\| L_{\sigma,j}^{(i_a)}(\mathbf{x}) \right\|_{\sigma}}^{=\mathcal{O}(1)} \\
&= \frac{\mathcal{O}(1)}{\left| N_{K/\mathbb{Q}} \left((\beta_1^{(i_a)})^{n_{i_a}} \right) \right|^{d_{i_a}-1} \prod_{i \in I_a} \left| N_{K/\mathbb{Q}} \left((\beta_1^{(i)})^{n_i} \right) \right|} \times \prod_{j=1}^{d_{i_a}} \left(\prod_{m=2}^{d_{i_a}} \left(\left| \beta_m^{(i_a)} \right|^{n_{i_a}} \right)^{\frac{|\Gamma|}{d_{i_a}}} \right) \\
&= \mathcal{O} \left(\frac{1}{\left| (\beta_1^{(i_a)})^{n_{i_a}} \right|^{|\Gamma|} \prod_{\substack{i \in I_a \\ i \neq i_a}} \left| N_{K/\mathbb{Q}} \left((\beta_1^{(i)})^{n_i} \right) \right|} \right)
\end{aligned} \tag{21}$$

Finally let us consider the case $i = i_0$ and similarly as before we obtain by (18) and (19) the inequality

$$\begin{aligned}
\prod_{j=1}^{d_{i_0}} \prod_{\nu \in S} \left| L_{\nu,j}^{(i_0)}(\mathbf{x}) \right|_{\nu} &= \prod_{j=1}^{d_{i_0}} \prod_{\nu \in S_{\text{fin}}} \left| L_{\nu,j}^{(i_0)}(\mathbf{x}) \right|_{\nu} \\
&\times \prod_{j=1}^{d_{i_0}} \prod_{m=2}^{d_{i_0}} \prod_{\substack{\sigma \in \Gamma \\ \sigma(\beta_j^{(i_0)}) = \beta_m^{(i_0)}}} \left\| L_{\sigma,j}^{(i_0)}(\mathbf{x}) \right\|_{\sigma} \times \prod_{j=1}^{d_{i_0}} \prod_{\substack{\sigma \in \Gamma \\ \sigma(\beta_j^{(i_0)}) = \beta_1^{(i_0)}}} \left\| L_{\sigma,j}^{(i_0)}(\mathbf{x}) \right\|_{\sigma} \\
&= \frac{\mathcal{O}(1)}{\left| N_{K/\mathbb{Q}} \left((\beta_1^{(i_0)})^{n_{i_0}} \right) \right|^{d_{i_0}-1} \prod_{i \in J} \left| N_{K/\mathbb{Q}} \left((\beta_1^{(i)})^{n_i} \right) \right|} \times \prod_{j=1}^{d_{i_0}} \left(\prod_{m=2}^{d_{i_0}} \left(\left| \beta_m^{(i_0)} \right|^{n_{i_0}} \right)^{\frac{|\Gamma|}{d_{i_0}}} \right) \\
&\times \prod_{j=1}^{d_{i_0}} \prod_{\substack{\sigma \in \Gamma \\ \sigma(\beta_j^{(i_0)}) = \beta_1^{(i_0)}}} \left| c_1^{(i_0)} \right| \cdot \left| \beta_1^{(i_0)} \right|^{n_{i_0}} \\
&= \mathcal{O} \left(\frac{\left| c_1^{(i_0)} \right|^{|\Gamma|}}{\prod_{\substack{i \in I_0 \\ i \neq i_0}} \left| N_{K/\mathbb{Q}} \left((\beta_1^{(i)})^{n_i} \right) \right|} \right)
\end{aligned} \tag{22}$$

Combining the estimates (20), (21) and (22) we obtain

$$\prod_{\nu \in S} \prod_{i,j} \left| L_{\nu,j}^{(i)}(\mathbf{x}) \right|_{\nu} = \mathcal{O} \left(\frac{|c_1^{(i_0)}|^{| \Gamma |}}{\prod_{\substack{i \in I_0 \cup I_a, \\ i \neq i_0}} \left(|\beta_1^{(i)}|^{n_i} \right)^{| \Gamma |}} \right). \quad (23)$$

5.3. Proof of Proposition 1. In the last subsection we estimated the product over the linear forms we chose to apply the Subspace Theorem (Theorem 3). In order to apply the Subspace Theorem we have to compute $\overline{|\mathbf{x}|}$. Since $N - A_i = x_1^{(i)} + \dots + x_{d_i}^{(i)}$ and $x_j^{(i)} = \mathcal{O}(1)$ unless $j = 1$ by Lemma 4, we deduce that $x_1^{(i)} = N + \mathcal{O}(1)$ for all $1 \leq i \leq s$. By Lemma 3 all the $x_j^{(i)}$ are conjugated, hence $\overline{|\mathbf{x}|} = N + \mathcal{O}(1)$.

Let us consider (23) a bit closer. Note that $c_1^{(i)} \left(\beta_1^{(i)} \right)^{n_i} = N + \mathcal{O}(1)$. Then we obtain

$$\prod_{\nu \in S} \prod_{i,j} \left| L_{\nu,j}^{(i)}(\mathbf{x}) \right|_{\nu} = \mathcal{O} \left(\frac{\prod_{i \in I_{0,a}} |c_1^{(i)}|^{| \Gamma |}}{\prod_{\substack{i \in I_{0,a} \\ i \neq i_0}} |c_1^{(i)}| \left(\beta_1^{(i)} \right)^{n_i}} \right) = \mathcal{O} \left(\frac{\prod_{i \in I_{0,a}} |c_1^{(i)}|^{| \Gamma |}}{N^{(s-1)| \Gamma |}} \right)$$

First, let us assume that $\prod_{i \in I_{0,a}} |c_1^{(i)}| > N^{s-1-\epsilon/| \Gamma |}$. Thus in view of inequality (23) we have

$$\frac{|c_1^{(i_0)}|}{\prod_{\substack{i \in I_{0,a} \\ i \neq i_0}} |\beta_1^{(i)}|^{n_i}} \geq N^{-\epsilon}$$

or equivalently

$$CN^{1+\epsilon} > |c_1^{(i_0)}| \cdot |\beta_1^{(i_0)}|^{n_i} N^{\epsilon} \geq \prod_{i \in I_{0,a}} |\beta_1^{(i)}|^{n_i}$$

for some constant C which yields Proposition 1.

Therefore we may assume that $\prod_{i \in I_{0,a}} |c_1^{(i)}| < N^{s-1-\epsilon/| \Gamma |}$ for some $\epsilon > 0$. Then we obtain

$$\prod_{\nu \in S} \prod_{i,j} \left| L_{\nu,j}^{(i)}(\mathbf{x}) \right|_{\nu} < \frac{C}{N^{\epsilon}} < N^{-\epsilon/2} < \overline{|\mathbf{x}|}^{-\epsilon/3}, \quad (24)$$

provided N is sufficiently large and C denotes some (sufficiently large) constant. Since D is the common denominator of the $c_j^{(i)}$ with $1 \leq i \leq s$, $1 \leq j \leq d_i$ and since the A_i for $1 \leq i \leq s$ are bounded by a constant (cf. Lemma 4), inequality (24) still holds, if we replace \mathbf{x} by $D\mathbf{x}$. Thus by the Subspace Theorem we obtain that all solutions \mathbf{x} subject to (15) lie on finitely many proper subspaces T . But, we can show that these subspaces contain only bounded solution and in particular there exist at most finitely many exceptional N . In particular, the proof of the Proposition is complete by proving the following lemma:

Lemma 5. *All solutions \mathbf{x} subject to (15) that lie on a proper subspace T are bounded, i.e. $\overline{|\mathbf{x}|} < C$ for some constant C .*

Proof. In order to prove Lemma 5 it is sufficient to prove that only bounded solutions \mathbf{x} satisfying (15) lie on a proper linear subspace $T \subset K^{|X|}$, i.e. satisfy a homogenous equation of the form

$$t_1^{(i_0)} x_1^{(i_0)} + t_1^{(i_a)} x_1^{(i_a)} + \sum_{i \in I_{0,a}} \sum_{j=2}^{d_i} t_j^{(i)} x_j^{(i)} = 0, \quad (25)$$

where $|x_j^{(i)}| = \mathcal{O}(1)$ unless $j = 1$ and not all $t_j^{(i)}$ vanish. In case that $a = 0$ we immediately obtain $|x_1^{(i_0)}| = \mathcal{O}(1)$, hence N is bounded and therefore also $|\overline{\mathbf{x}}|$.

In case that $a \neq 0$ we consider the linear system

$$\begin{aligned} 0 &= t_1^{(i_0)} x_1^{(i_0)} + t_1^{(i_a)} x_1^{(i_a)} + \overbrace{\sum_{i \in I_{0,a}} \sum_{j=2}^{d_i} t_j^{(i)} x_j^{(i)}}^{=\mathcal{O}(1)} \\ a &= x_1^{(i_0)} - x_1^{(i_a)} + \overbrace{\sum_{j=2}^{d_i} x_j^{(i_0)} - x_j^{(i_a)}}^{=\mathcal{O}(1)}. \end{aligned} \quad (26)$$

Let us assume for the moment that $t_1^{(i_a)} \neq -t_1^{(i_0)}$. If we add $t_1^{(i_a)}$ times the second equation to the first equation in the linear system (26), then we see that $x_1^{(i_0)}$ can be written as the linear combination of the $x_j^{(i)}$ with $j > 1$, i.e. $x_1^{(i_0)}$ is bounded and N and $|\overline{\mathbf{x}}|$ are bounded too.

Therefore we may assume that $t_1^{(i_a)} = -t_1^{(i_0)}$ and the linear system (26) turns into

$$\sum_{i \in I_{0,a}} \sum_{j=2}^{d_i} u_j^{(i)} x_j^{(i)} = b \quad (27)$$

where $u_j^{(i_0)} = t_j^{(i_0)} - t_1^{(i_0)}$, $u_j^{(i_a)} = t_j^{(i_a)} - t_1^{(i_a)}$, $u_j^{(i)} = t_j^{(i)}$ if $i \neq i_0, i_a$ and $b = at^{(i_a)}$. Let us recall that the Galois group of K is of the form $\Gamma = \prod_{i \in I_{0,a}} \Gamma^{(i)}$. Let us take some $\sigma \in \Gamma$ such that $\sigma(\beta_m^{(\ell)}) = \beta_1^{(\ell)}$ and $\sigma|_{K^{(i)}} = \text{id}$ for all $i \neq \ell$. If we apply σ to equation (27) we obtain

$$\begin{aligned} \sigma(u_m^{(\ell)}) \sigma(x_m^{(\ell)}) + \sum_{i \in I_{0,a}} \sum_{\substack{2 \leq j \leq d_i \\ (i,j) \neq (\ell,m)}} \sigma(u_j^{(i)}) \sigma(x_j^{(i)}) = \\ \sigma(u_m^{(\ell)}) x_1^{(\ell)} + \overbrace{\sum_{i \in I_{0,a}} \sum_{j=2}^{d_i} \tilde{u}_j^{(i)} x_j^{(i)}}^{\mathcal{O}(1)} = \sigma(b), \end{aligned}$$

where $\tilde{u}_j^{(i)} = \sigma(u_k^{(i)})$ where k satisfies $\sigma(\beta_k^{(i)}) = \beta_j^{(i)}$. Therefore either $\sigma(u_m^{(\ell)}) = 0$, i.e. $u_m^{(\ell)} = 0$, or $|x_1^{(\ell)}| = \mathcal{O}(1)$. Therefore varying m and ℓ over all possibilities we obtain either $u_i^{(j)} = 0$ for all admissible pairs (i, j) or the solution \mathbf{x} is bounded. Therefore we may assume that all $t_j^{(i_0)} = t_1^{(i_0)}$, $t_j^{(i_a)} = t_1^{(i_a)} = -t_1^{(i_0)}$ and $t_j^{(i)} = 0$ if

$i \neq i_0, i_a$. Furthermore, we may assume that $t_1^{(i_0)} \neq 0$ since otherwise all $t_j^{(i)}$ would vanish which contradicts the fact that T is a proper subspace. Therefore the linear system (26) yields

$$0 = x_1^{(i_0)} - x_1^{(i_a)} + \sum_{j=2}^{d_i} x_j^{(i_0)} - x_j^{(i_a)} = a \neq 0,$$

a contradiction. \square

Remark 4. We want to emphasize that the only point where we used the assumption that the $K^{(i)}$ are linearly disjoint over \mathbb{Q} is in the proof of Lemma 5. If we do not assume that the $K^{(i)}$ are linearly disjoint, then we cannot assure that certain coefficients $t_j^{(i)}$ in (25) vanish and the proof breaks down.

Remark 5. In view of Lemma 5 the constant $C_{\varepsilon, \beta}$ in Theorem 2 depends on the coefficients $t_j^{(i)}$ from the equation defining the subspace T . Since the ineffectiveness of the absolute Subspace Theorem also the constant $C_{\varepsilon, \beta}$ remains ineffective.

REFERENCES

- [1] M. Drmota and R. F. Tichy. *Sequences, discrepancies and applications*, volume 1651 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1997.
- [2] J.-H. Evertse and R. G. Ferretti. A further improvement of the quantitative subspace theorem. *Ann. of Math. (2)*, 177(2):513–590, 2013.
- [3] J.-H. Evertse and H. P. Schlickewei. A quantitative version of the absolute subspace theorem. *J. Reine Angew. Math.*, 548:21–127, 2002.
- [4] A. S. Fraenkel. Systems of numeration. *Amer. Math. Monthly*, 92(2):105–114, 1985.
- [5] C. Frougny and B. Solomyak. Finite beta-expansions. *Ergodic Theory and Dynamical Systems*, 12:713–723, 1992.
- [6] P. Grabner, P. Hellekalek, and P. Liardet. The dynamical point of view of low-discrepancy sequences. *Uniform Distribution Theory*, 7(1):11–70, 2012.
- [7] P. Grabner, P. Liardet, and R. Tichy. Odometers and systems of numeration. *Acta Arithmetica*, 70:103–123, 1995.
- [8] J. Hartinger, R. Kainhofer, and V. Ziegler. On the corner avoidance properties of various low-discrepancy sequences. *Integers*, 5(3):16pp, 2005.
- [9] J. Hartinger and V. Ziegler. On corner avoidance properties of random-start Halton sequence. *SIAM J. Numer. Anal.*, 45(3):1109–1121, 2007.
- [10] E. Hlawka. Funktionen von beschränkter Variation in der Theorie der Gleichverteilung. *Annali di Matematica Pura ed Applicata*, (54):325–333, 1961.
- [11] M. Hofer, M. Iacò, and R. Tichy. Ergodic properties of β -adic Halton sequences. *to appear in Ergodic Theory and Dynamical Systems*.
- [12] J. Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.
- [13] A. Owen. Halton sequences avoid the origin. *SIAM Rev.*, 48(3):487–503, 2006.
- [14] H. P. Schlickewei. The p -adic Thue-Siegel-Roth-Schmidt theorem. *Arch. Math. (Basel)*, 29(3):267–270, 1977.
- [15] W. M. Schmidt. Linear forms with algebraic coefficients. I. *J. Number Theory*, 3:253–277, 1971.
- [16] W. M. Schmidt. *Diophantine approximations and Diophantine equations*, volume 1467 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1991.
- [17] I. M. Sobol. On the distribution of points in a cube and the approximate evaluation of integrals. *USSR Comput. Math. Math. Phys.*, 7:86–112, 1967.
- [18] B. Solomyak. Substitutions, adic transformations and beta-expansions. *in: Contemporary Mathematics (Proc. of the Conference on Symbolic Dynamics, Yale, 1991)*, 135:361–372, 1992.

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